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Higher-Order Recurrence in Tachibana Spaces and Curvature Compatibility with Weyl-Tachibana and H-Concircular Structures

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Abstract:
This study explores the geometric structure of Tachibana spaces equipped with higher-order recurrent curvature tensors. Building upon foundational concepts in almost Hermitian and Kahlerian geometry, we define and investigate r-recurrent Tachibana spaces (rT^n-spaces) and analyze the associated recurrence conditions in Weyl-Tachibana and Tachibana H-concircular curvature tensors, denoted respectively as Z-rT^n and E-rT^n spaces. We establish necessary and sufficient conditions under which these recurrence structures are mutually equivalent or imply one another. Furthermore, the study reveals the interdependence between recurrence in various curvature tensors, contributing to a more unified understanding of higher-order geometric structures. The findings extend classical results in Kahlerian geometry and provide potential applications in differential geometry, mathematical physics, and symmetric theoretical models.

Keyword: Tachibana space, r-recurrence, recurrent curvature tensor, Weyl-Tachibana tensor, H-concircular tensor.

1. Introduction:

1.1. Overview:

The study of recurrent geometric structures in differential geometry offers deep insight into the underlying symmetry and invariance properties of spaces. Tachibana spaces, as a generalization of Kahlerian spaces, provide a framework where complex structure, Hermitian metrics, and curvature behaviors interact in intricate ways. In this work, we investigate the phenomenon of higher-order recurrence in Tachibana spaces and explore how it manifests in various associated curvature tensors, such as Weyl-Tachibana and H-concircular tensors.

1.2. Historical Background:

The notion of recurrence in Riemannian and Hermitian geometry dates back to seminal works by Walker and Yano, who first examined recurrent curvature structures and their symmetries.
Subsequent contributions by Mathai, Negi, Singh, and Nautiyal extended these ideas into Kahlerian and complex geometric frameworks. The concept of Tachibana spaces, which broadens the classical Kahlerian structure, enabled the study of complex spaces with more flexible curvature conditions and symmetry constraints. However, despite these advances, the characterization of r-recurrence in Tachibana geometry remains largely unexplored.

1.3. Research Gap:

While various studies have focused on recurrent curvature in Kahlerian and almost Hermitian spaces, a comprehensive analysis of higher-order recurrence (r-recurrence) within Tachibana spaces is still lacking. Furthermore, the interrelation between the recurrence of the main curvature tensor and that of the Weyl-Tachibana and H-concircular curvature tensors has not yet been systematically examined. Most existing works treat these structures independently without exploring their compatibility or equivalence under recurrence.

1.4. Literature Review:

Key contributions in the domain include:
- Singh and Kumar [6,11]: Introduced semi-recurrence in Kahler spaces and preliminary notions of r-recurrence.
- Negi [2,4]: Investigated conharmonic and H-birecurrent structures, primarily within the Kahler framework.
- Yano [8]: studied the role of the Nijenhuis tensor in complex geometry, laying the groundwork for recurrence studies in Hermitian structures.
- Singh and Kothari [12]: Undertook a rigorous and systematic analysis of the Tachibana concircular curvature tensor, conceptualized as the Kahlerian analogue of the classical concircular curvature tensor in Riemannian geometry. Their work significantly contributes to the advancement of differential geometric theory by enhancing the understanding of specialized curvature tensors within the broader context of complex spaces.

These efforts collectively underscore the importance of integrating recurrence studies across different curvature tensors within broader Hermitian geometries.

1.5. Objectives and Contributions:

1.5.1. Objectives:

The primary objectives of this research are to:
1. Formulate the concept of r-recurrence in Tachibana spaces (rT^n-spaces).
2. Define and analyze recurrence in Weyl-Tachibana Z-rT^n and H-concircular E-rT^n curvature tensors.
3. Establish conditions under which recurrence in one tensor implies recurrence in another.
4. Investigate the interdependence and structural compatibility among different curvature tensors under higher-order recurrence.

1.5.2. Contributions:

The major contributions of this work include:
- A formal definition and analytical framework for r-recurrent Tachibana spaces and their associated curvature tensors.
- The establishment of necessary and sufficient conditions for equivalence and implication among R-rT^n, Z-rT^n, and E-rT^n structures.
- A unifying perspective that integrates different recurrent structures into a coherent geometrical model.
- A generalization of classical Kahlerian recurrence results in the broader category of Tachibana geometry.

2. Preliminaries on Tachibana Spaces and Recurrent Properties:

The geometry of Kahlerian spaces has been extensively studied, with foundational contributions from Mathai [9] and Walker [7], who analyzed Ruse-type spaces with recurrent curvature tensors. Singh and Nautiyal [11] advanced the theory by introducing various recurrence properties in Kahler spaces, leading to significant structural results. Building on this foundation, Singh and Kumar [6] extended these investigations to Tachibana spaces, deriving notable recurrence conditions and generalizations.
In this section, we present the fundamental definitions and structural identities associated with Tachibana spaces that form the basis for our study of higher-order recurrence.

2.1. Definitions and Basic Properties of Tachibana Spaces:

An almost Tachibana space is defined as an almost Hermitian space (F_i^h,g_{ij}), where F_i^h represents an almost complex structure, and g_{ij} is a Hermitian metric. These satisfy the condition

$$F_{i,j}^h + F_{j,i}^h = 0, \qquad (1)$$

where the comma denotes covariant differentiation with respect to the symmetric connection Γ_{ij}^h. In an almost Tachibana space, the Nijenhuis tensor, denoted by N_{ji}^h, is given by Yano [8] as

$$N_{ji}^h = -4(F_{i,j}^a)F_a^h, \qquad (2)$$

Here, $F_{i,j}^h$ is purely skew-symmetric, and N_{ji}^h serves as a measure of the nonintegrability of the structure. If $N_{ji}^h = 0$, the space is called a Tachibana space, denoted as a T^n-space.

2.2. Recurrent Tachibana Spaces:

A Tachibana space is termed recurrent if its curvature tensor R_{ijk}^h satisfies[1,10]:

$$R_{ijk,a}^h = \lambda_a R_{ijk}^h, \qquad (3)$$

$$\text{or,} \qquad R_{ijk,a}^h - \lambda_a R_{ijk}^h = 0,$$

where λ_a is a non-zero recurrence vector field. The curvature tensor R_{ijk}^h is explicitly defined as

$$R_{ijk}^h = \frac{\partial\Gamma_{ik}^h}{\partial x^j} - \frac{\partial\Gamma_{ij}^h}{\partial x^k} + \Gamma_{ik}^m\Gamma_{mj}^h - \Gamma_{ij}^m\Gamma_{mk}^h. \qquad (4)$$

The associated Ricci tensor and scalar curvature are given by $R_{ij} = R_{ijh}^h$ and $R= R_{ij}g^{ij}$.

2.3. Higher-Order Recurrence in Tachibana Spaces:

A Tachibana r-recurrent space (denoted as an rT^n-space) satisfies

$$R_{ijk,a_1a_2---a_r}^h = \lambda_{a_1a_2---a_r} R_{ijk}^h, \qquad (5)$$

where $\lambda_{a_1a_2---a_r}$ is a non-zero recurrence tensor field.
Similarly, a Tachibana Ricci-r recurrent space (denoted as an R-rT^n-space) satisfies

$$R_{ij,a_1a_2---a_r} = \lambda_{a_1a_2---a_r} R_{ij} \qquad (6)$$

Multiplying (6) by g^{ij} and using the fact that $g_{,a_1a_2---a_r}^{ij} = 0$, we obtain

$$R_{,a_1a_2---a_r} = \lambda_{a_1a_2---a_r} R. \qquad (7)$$

2.4. Curvature Tensors in Tachibana Spaces:

The Weyl-concircular curvature tensor in a T^n-space is defined as [3]:

$$Z_{ijk}^h = R_{ijk}^h + \frac{R}{n(n-1)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h). \qquad (8)$$

Similarly, the Tachibana H-concircular curvature tensor is given by [5]:

$$E_{ijk}^h = R_{ijk}^h + \frac{R}{n(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h). \qquad (9)$$

From these expressions, we derive

$$E_{ijk}^h = Z_{ijk}^h - \frac{3R}{n(n-1)(n+2)}(g_{ik}\delta_j^h - g_{jk}\delta_i^h) + \frac{R}{n(n+2)}(F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h). \qquad (10)$$

2.5. Remark:

It is evident from equation (5) that every rT^n-space is necessarily an R-rT^n-space. However, the converse does not hold in general, as Ricci recurrence does not imply the recurrence of the full curvature tensor. This distinction forms the basis for further investigation into the relationships between recurrence conditions in different geometric structures.

3. Weyl–Tachibana and Tachibana H–Concircular r-Recurrent Structures:

In this section, we introduce and investigate higher-order recurrence conditions associated with the Weyl–Tachibana and Tachibana H–concircular curvature tensors. We define their respective recurrence structures and establish interdependencies with standard curvature recurrence in Tachibana spaces. These results contribute to the broader theory of structural compatibility in complex differential geometry.

3.1. Definition: Weyl-Tachibana Concircular r-Recurrent Spaces:

A Tachibana space T^n is said to be Weyl-Tachibana concircular r-recurrent if its Weyl-Tachibana curvature tensor Z^h_{ijk} satisfies the recurrence condition:

$$Z^h_{ijk,a_1a_2---a_r} = \lambda_{a_1a_2---a_r} Z^h_{ijk}, \qquad (11)$$

where $\lambda_{a_1a_2---a_r}$ is a non-vanishing symmetric covariant tensor of order r. Such a space is referred to as a Z-rT^n-space.

3.2. Definition: Tachibana H–Concircular r-Recurrent Spaces:

A Tachibana space T^n is said to be Tachibana H-concircular r-recurrent if its H-concircular curvature tensor E^h_{ijk} satisfies:

$$E^h_{ijk,a_1a_2---a_r} = \lambda_{a_1a_2---a_r} E^h_{ijk}, \qquad (12)$$

for some non-zero symmetric covariant tensor field $\lambda_{a_1a_2---a_r}$. A space satisfying this condition is termed an E-rT^n-space.

Theorem 3.1:

Let T^n be an rT^n-space. Then the Weyl-Tachibana concircular curvature tensor also satisfies the r-recurrence condition. Hence, every rT^n-space is necessarily a Z-rT^n-space.

Proof:

By differentiating the definition of the Weyl-Tachibana concircular curvature tensor, we obtain:

$$Z^h_{ijk,a_1a_2---a_r} = R^h_{ijk,a_1a_2---a_r} + \frac{R,_{a_1a_2---a_r}}{n(n-1)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \qquad (13)$$

Multiplying the recurrence condition by $\lambda_{a_1a_2---a_r}$ and subtracting, we get:

$$Z^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} Z^h_{ijk} = R^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} R^h_{ijk} + \frac{R,_{a_1a_2---a_r} - \lambda_{a_1a_2---a_r}R}{n(n-1)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \qquad (14)$$

Since T^n is an rT^n-space, recurrence conditions hold, reducing this to

$$Z^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} Z^h_{ijk} = 0,$$

Thus, T^n is a Z-rT^n-space, completing the proof. Which shows that the space is Z-rT^n-space.

Theorem 3.2:

If a Tachibana space T^n is r-recurrent, then its Tachibana H-concircular curvature tensor also satisfies an r-recurrence condition. Consequently, every rT^n-space is an E-rT^n-space.

Proof:

Differentiating the definition of the Tachibana H-concircular curvature tensor, we obtain:

$$E^h_{ijk,a_1a_2---a_r} = R^h_{ijk,a_1a_2---a_r} + \frac{R,_{a_1a_2---a_r}}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \qquad (15)$$

Multiplying the recurrence condition by $\lambda_{a_1a_2---a_r}$ and subtracting, we get:

$$E^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} E^h_{ijk} = R^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} R^h_{ijk} + \frac{R,_{a_1a_2---a_r} - \lambda_{a_1a_2---a_r}R}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \qquad (16)$$

Since T^n is an rT^n-space, the recurrence conditions hold, simplifying this to

$$E^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} E^h_{ijk} = 0,$$

Thus, T^n is an E-rT^n-space, completing the proof.

Theorem 3.3:

In a Tachibana space T^n, if any two of the following recurrence conditions hold:

1. The space is an R-rT^n-space
2. The space is a Z-rT^n-space,
3. The space is an E-rT^n-space,

then the third condition is necessarily satisfied.

Proof:

Differentiating equation (10) with respect to the recurrence parameters, we obtain:

$$E^h_{ijk,a_1a_2---a_r} = Z^h_{ijk,a_1a_2---a_r} - \frac{3R,_{a_1a_2---a_r}}{n(n-1)(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \frac{R,_{a_1a_2---a_r}}{n(n+2)}(F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \quad (17)$$

Multiplying equation (10) by $\lambda_{a_1a_2---a_r}$ and subtracting the resulting expression from the above equation (17), we get:

$$E^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} E^h_{ijk} = Z^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} Z^h_{ijk} - \frac{3(R,_{a_1a_2---a_r} - \lambda_{a_1a_2---a_r}R)}{n(n-1)(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \frac{(R,_{a_1a_2---a_r} - \lambda_{a_1a_2---a_r}R)}{n(n+2)}(F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \qquad (18)$$

Utilizing equations (7),(11),(12), and (18), we conclude that the given recurrence conditions are interdependent, proving the theorem.

Theorem 3.4:

A Weyl-Tachibana concircular r-recurrent space (Z-rT^n-space) is an rT^n-space if and only if it is also an R-rT^n-space.

Proof:

Let the given Z-rT^n-space also be an rT^n-space. Then, by definition, equations (5) and (11) hold. Substituting these into equation (14), we obtain:

$$(R,_{a_1a_2---a_r} - \lambda_{a_1a_2---a_r}R) = 0.$$

This confirms that the space satisfies the condition of an R-rT^n-space.

Conversely, suppose the Z-rT^n-space is also an R-rT^n-space, which implies that equations (7) and (11) are satisfied. Substituting these into equation (14), we derive:

$$R^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} R^h_{ijk} = 0.$$

Thus, the space satisfies the definition of an rT^n-space, completing the proof.

Theorem 3.5:

A necessary and sufficient condition for an E-rT^n-space to be an rT^n-space is that the space is an R-rT^n-space.

Proof:

Let the given E-rT^n-space be an rT^n-space. In this case, equations (5) and (12) hold. By substituting these conditions into equation (16), we obtain

$$R_{,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} R = 0,$$

This confirms that the space satisfies the condition of an R-rT^n-space.

Conversely, suppose that the E-rT^n-space is also an R-rT^n-space. Then, equations (7) and (12) hold, and substituting these into equation (16) gives:

$$R^h_{ijk,a_1a_2---a_r} - \lambda_{a_1a_2---a_r} R^h_{ijk} = 0,$$

which shows that the space satisfies the definition of an rT^n-space. Thus, the proof is completed.

Theorem 3.6:

If any two of the properties:

- rT^n-space,
- R-rT^n-space,
- Z-rT^n-space

hold in a Tachibana space, the third follows necessarily.

Proof:

By utilizing equations (5),(7), and (14), the interrelationship between the properties is straightforward to verify. The completion of the proof follows directly from the established relationships between these conditions.

Theorem 3.7:

If any two of the following conditions hold in a Tachibana space:

- rT^n-space,
- R-rT^n-space,
- E-rT^n-space

then the third condition is automatically satisfied.

Proof:

Using equations (5),(7), and (16), the interdependence of the properties can be easily demonstrated. The proof follows immediately from the established relationships among these properties, thus concluding the theorem.

4. Applications and Implications in Geometry and Mathematical Physics:

The results derived in this study have noteworthy applications across both theoretical and applied domains, particularly in areas where geometric structures and curvature conditions are essential.

- Mathematical Physics: The investigation of higher-order recurrence properties in Tachibana spaces offers meaningful contributions to general relativity and string theory. In these frameworks, curvature symmetries are directly related to conservation laws and invariant physical quantities, making recurrent structures highly relevant in modeling space-time geometry and field equations.
- Geometric Analysis: The established compatibility among various curvature tensors provides powerful tools for addressing nonlinear geometric partial differential equations and analyzing geometric flows. This is particularly valuable in studying long-time behavior, singularity formation, and curvature-driven evolution equations.
- Computational Geometry and Robotics: Recurrence conditions on curvature tensors assist in designing algorithms for motion planning on curved surfaces and spaces. These insights are beneficial in robotic systems constrained by symmetry, where trajectory optimization and geometric path following require an understanding of the underlying space structure.
- Scientific Computing and Data Repositories: In the context of scientific portals and computational geometry platforms, the derived results may be integrated into classification algorithms or simulation tools that model spaces with recurrent curvature features. These can serve as benchmarks or standard models in digital geometry processing and geometric machine learning.

5. Conclusion:

This paper offers a comprehensive and rigorous examination of higher-order recurrence properties in Tachibana spaces. By introducing precise definitions and systematically deriving key relationships, we have demonstrated that r-recurrence in the fundamental curvature tensor leads directly to corresponding recurrence in both the Weyl-Tachibana and Tachibana H-concircular curvature tensors.

Among the central contributions, Theorem 2.2 establishes that any r-recurrent Tachibana space necessarily induces r-recurrence in its H-concircular curvature tensor. This fundamental result confirms that every rT^n-space is also an E-rT^n space, thereby revealing a strong structural compatibility between these curvature constructs.

The series of theorems developed in this work further unveils deep geometric interdependencies among R-rT^n, Z-rT^n, and E-rT^n spaces. These interrelationships reinforce the internal coherence of recurrence properties and enrich the broader theoretical framework surrounding Tachibana geometry.

Collectively, the results contribute to a unified understanding of recurrence phenomena in differential geometry, particularly within the contexts of Hermitian and almost Hermitian manifolds. The theoretical advancements outlined here hold potential for future research in geometric analysis, complex geometry, and mathematical physics-particularly in domains where recurrent structures and curvature symmetries play a fundamental role.

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